

Lagrangian and Hamiltonian formalism in Field Theory: a simple model

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Abstract

The static of smooth maps from the two-dimensional disc to a smooth manifold can be regarded as a simplified version of the Classical Field Theory. In this paper we construct the Tulczyjew triple for the problem and describe the Lagrangian and Hamiltonian formalism. We outline also natural generalizations of this approach to arbitrary dimensions.

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1 Introduction

The main purpose of this work is to implement the Tulczyjew triple approach of the Analytical Mechanics [26, 27] into the statics of multi-dimensional objects, i.e. smooth maps from a disc $D \subset \mathbb{R}^n$ into a manifold M . This problem can be regarded as a toy model for the Classical Field Theory, since the set of smooth maps from \mathbb{R}^n to M can be treated as a set of sections of the trivial bundle $pr_1 : \mathbb{R}^n \times M \rightarrow \mathbb{R}^n$. In comparison with general geometric approaches [3, 4, 25] the situation is considerably simplified, because the bundle is trivial and the base manifold \mathbb{R}^n has a canonical volume form and a canonical base of sections of the tangent bundle. For $n = 1$ and M being the space of configurations of a mechanical system we recover the model of the autonomous mechanics.

We work with this geometrically simple version of the Classical Field Theory to present the main ideas of our approach to the Lagrangian and Hamiltonian formalism that differs from the ones which are present in the literature [15]. Since we skipped topological difficulties in this case, we could concentrate on the recognition of physically important objects, like the phase space, phase dynamics, Legendre map, Hamiltonian, etc. These issues are usually not elaborated well in the literature, as the Classical Field Theory models use to concentrate on the Euler-Lagrange equations. Of course, we recover also the commonly accepted Euler-Lagrange equations, this time without requiring any regularity of the Lagrangian.

The methods we use are based on expressing the theory in terms of differential relations rather than maps or tensor fields. For the price of dealing with differential calculus of relations we get, in our opinion, better understanding of geometric structures involved. It was also shown in [10, 8] that using the same philosophy one can pass easily to the more complicated geometrical framework based on Lie or general algebroids. In the case of Analytical Mechanics similar generalizations were proposed by many authors (e.g. [19, 20, 17]), but the approach presented in [10, 8], being ideologically simpler, will be our starting point.

We would like to point out that all the constructions we perform are motivated by the variational calculus that we consider to be the fundamental idea of Classical Mechanics and Field Theory. The origin of geometric structures we use lies in the rigorous formulation of the variational principle including boundary terms that one can find in [25, 22]. Nevertheless, we do not enter into details of the

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variational calculus and we treat it rather as a guide-line for recognizing which geometrical structures are appropriate in this case.

The problem itself, i.e. the generalization of the symplectic framework for autonomous mechanics to higher dimensions is not new and was first treated by Günther in [12]. The underlying geometric structure of Günther's theory, known as k -symplectic structure, was described systematically in [1, 2]. Recently, Rey, Roman-Roy, Salgado and Valarino renewed the theory and described its Lie algebroid version [24]. Our work is also related to the multisymplectic approach to the Classical Field Theory developed by Gotay, Isenberger, Marsden and others and presented e.g. in [3, 4, 5, 6]. The Tulczyjew triple in the context of multisymplectic field theories appeared already in [18].

For the presentation of our general idea, let us first recall the description of the dynamics of a classical autonomous mechanical system without constraints. Let M denote the manifold of positions of the system. The trajectory is therefore a smooth path in M , i.e. a map from the time interval $[t_0, t_1] \subset \mathbb{R}$ into M . We can try to describe our system in variational way, looking for those trajectories $\gamma : \mathbb{R} \rightarrow M$ that, for the fixed time interval $[t_0, t_1]$, minimize the action functional

$$(1.1) \quad S(\gamma) = \int_{t_0}^{t_1} L(\mathbf{t}\gamma(t)) dt.$$

We assumed above that the Lagrangian is of first-order i.e. it is a function on the tangent bundle TM . The curve $t \mapsto \mathbf{t}\gamma(t)$ will denote the *tangent prolongation* of the curve γ in M . The variational approach for the finite time interval leads to the Euler-Lagrange equations and the definition of momenta. The space of momenta is usually called the *phase space* of the system. In the case of autonomous mechanics, the phase space is just the cotangent bundle T^*M .

We describe the system by a first-order differential equation on the phase space, called the *phase dynamics*. The phase dynamics \mathcal{D} is described by a subset of TT^*M :

$$(1.2) \quad \mathcal{D} = \alpha_M^{-1}(dL(TM)),$$

where α_M is the Tulczyjew isomorphism (defined in [26]) $\alpha_M : TT^*M \rightarrow T^*TM$ and $dL(TM)$ is the image of the differential of the Lagrangian. A curve $t \mapsto \eta(t) \in T^*M$ satisfies the phase dynamics if its tangent prolongation lies in \mathcal{D} . A curve $t \mapsto \gamma(t) \in TM$ satisfies, in turn, the corresponding Euler-Lagrange equation, if the curve $t \mapsto \alpha_M^{-1}(dL(\gamma(t))) \in TT^*M$ is the tangent prolongation of its projection to T^*M (see [10, 8]).

All the structures needed for generating the dynamics from the Lagrangian can be summarized in the following commutative diagram:

$$(1.3) \quad \begin{array}{ccccc} & TT^*M & \xrightarrow{\alpha_M} & T^*TM & \\ \tau_{T^*M} \swarrow & & & \searrow \zeta & \\ T^*M & \xrightarrow{id} & T^*M & \xrightarrow{\pi_{TM}} & TM \\ \pi_M \searrow & & \swarrow \pi_M & & \searrow \pi_M \\ & TM & \xrightarrow{id} & TM & \\ \tau_M \swarrow & & \searrow \tau_M & & \\ & M & \xrightarrow{id} & M & \end{array}$$

The Lagrangian formulation can also be obtained from the variational principle, when instead of finite domain of integration we use the so called de Rham current with one-point support (see [25]).

It may happen that the phase dynamics is an implicit differential equation, i.e. it is not the image of a vector field. In some cases, however, the phase dynamics is the image of a Hamiltonian vector field for some function $H : T^*M \rightarrow \mathbb{R}$. In such a case we can write:

$$(1.4) \quad \mathcal{D} = \beta_M^{-1}(dH(T^*M)),$$

where β_M is the canonical isomorphism between $\mathbb{T}\mathbb{T}^*M$ and $\mathbb{T}^*\mathbb{T}^*M$ given by the canonical symplectic form ω_M on \mathbb{T}^*M ,

$$\beta_M : \mathbb{T}\mathbb{T}^*M \longrightarrow \mathbb{T}^*\mathbb{T}^*M, \quad \langle \beta_M(v), w \rangle = \omega_M(v, w).$$

Let us recall for the future reference that the canonical symplectic form ω_M is defined by

$$(1.5) \quad \omega_M = d\vartheta_M,$$

where ϑ_M is the Liouville form given by

$$(1.6) \quad \vartheta_M(v) = \langle \tau_{\mathbb{T}^*M}(v), \mathbb{T}\pi_M(v) \rangle.$$

The structures needed for Hamiltonian mechanics can be presented in the following commutative diagram:

$$(1.7) \quad \begin{array}{ccccc} & & \mathbb{T}^*\mathbb{T}^*M & \xleftarrow{\beta_M} & \mathbb{T}\mathbb{T}^*M \\ & \swarrow \pi_{\mathbb{T}^*M} & & & \swarrow \tau_{\mathbb{T}^*M} \\ \mathbb{T}^*M & \xleftarrow{\xi} & \mathbb{T}^*M & \xleftarrow{id} & \mathbb{T}^*M \\ & \searrow \pi_M & & & \searrow \pi_M \\ & & \mathbb{T}M & \xleftarrow{id} & \mathbb{T}M \\ & \swarrow \tau_M & & & \swarrow \tau_M \\ M & \xleftarrow{id} & M & & M \end{array}$$

The formulation of the autonomous mechanics described above has at least two important features when compared with the ones in textbooks: it is very simple and can be easily generalized to more complicated cases including constraints, nonautonomous mechanics, and mechanics on algebroids [9, 10]. And last but not least: we need no regularity conditions for the Lagrangian. The Lagrangian can be a function, but it can be replaced by a family of functions generating a Lagrangian submanifold in $\mathbb{T}^*\mathbb{T}M$, as it happens in the case of a relativistic particle in the Minkowski space [28]. The crucial role is played by two mappings: α_M and β_M .

In what follows we replace ‘one dimensional’ objects, like time intervals and paths in a manifold M by ‘two dimensional’ objects, like discs and maps $u : \mathbb{R}^2 \rightarrow M$. We decided to keep $n = 2$ just for simplicity. However, generalization of our results to any natural n is straightforward.

We shall then find the phase space and the analog of the fundamental map α_M that allows us to obtain the phase equations from the Lagrangian. Then, we continue with the Hamiltonian formalism by recognizing what kind of a geometric object the Hamiltonian is, and by finding an analog of the map β_M .

2 Notation

Let M be a smooth manifold of dimension m . We denote by $\tau_M : \mathbb{T}M \rightarrow M$ the tangent vector bundle and by $\pi_M : \mathbb{T}^*M \rightarrow M$ the cotangent bundle of the manifold M . If $(q^a)_{a=1}^m$ is a local coordinate system in $U \subset M$, then we have the induced coordinate systems (q^a, \dot{q}^b) in $\tau_M^{-1}(U) \subset \mathbb{T}M$ and (q^a, p_b) in $\pi_M^{-1}(U) \subset \mathbb{T}^*M$. The above coordinates correspond to local sections (dq^b) of \mathbb{T}^*M and $(\frac{\partial}{\partial q^b})$ of $\mathbb{T}M$, respectively.

Let u be a smooth map from \mathbb{R}^2 to M . Since in the source space \mathbb{R}^2 we have two distinguished vector fields $\partial_{x^i} = \frac{\partial}{\partial x^i}$, $i = 1, 2$, the first jet $j^1u(x)$ of the mapping u at a point $x = (x^1, x^2)$ can be identified with a pair of vectors tangent to M at the point $u(x)$, i.e.

$$j^1u(x) = (v_1, v_2), \quad \text{where} \quad v_i = (\mathbb{T}_xu)(\partial_{x^i}) \in \mathbb{T}_{u(x)}M.$$

Therefore the set $J_x(\mathbb{R}^2, M)$ of the first jets of maps $\mathbb{R}^2 \rightarrow M$ at $x = (x^1, x^2) \in \mathbb{R}^2$ will be identified with

$$\overset{2}{\mathbb{T}}M = \mathbb{T}M \times_M \mathbb{T}M$$

and the element of $\overset{2}{\mathbb{T}}M$, corresponding to u at x , will be denoted by $\overset{2}{\mathfrak{t}}u(x)$. The manifold $\overset{2}{\mathbb{T}}M$ has a natural bundle structure over M :

$$\overset{2}{\mathbb{T}}M \ni \overset{2}{\mathfrak{t}}u(x) \longmapsto u(x) \in M.$$

The above projection will be denoted by τ_M^2 . Like for the tangent bundle, we have the adapted set of local coordinates $(q^a, \dot{q}_1^b, \dot{q}_2^c)$ in $(\tau_M^2)^{-1}(U) \subset \overset{2}{\mathbb{T}}M$. The bundle τ_M^2 is a vector bundle. Its dual $(\overset{2}{\mathbb{T}}M)^*$ can be identified with

$$\overset{2}{\mathbb{T}}^*M = \mathbb{T}^*M \times_M \mathbb{T}^*M$$

with the obvious pairing with $\overset{2}{\mathbb{T}}M$. The projection onto M in the dual bundle will be denoted by π_M^2 . We have also the adapted set of local coordinates (q^a, q_{b1}, p_{c2}) in $(\pi_M^2)^{-1}(U) \subset \overset{2}{\mathbb{T}}^*M$.

3 Variational approach

We start with Variational Calculus which is our guide-line for recognizing geometrical objects representing physical quantities. Let L be a smooth function on the manifold $\overset{2}{\mathbb{T}}M$ of the first jets of maps from $\mathcal{C}^\infty(\mathbb{R}^2, M)$; we will call L a Lagrangian. Any Lagrangian defines an action functional S on maps $u : D \rightarrow M$ from the unit disc $D \subset \mathbb{R}^2$ into M :

$$S(u) = \int_D L(u^a(x), \dot{u}_1^b(x), \dot{u}_2^c(x)) dx^1 dx^2,$$

where $u^a(x) = q^a(u(x))$ and $u_1^a(x) = \frac{\partial u^a}{\partial x^1}(x) = \dot{q}_1^a(\overset{2}{\mathfrak{t}}u(x))$, $u_2^a(x) = \frac{\partial u^a}{\partial x^2}(x) = \dot{q}_2^a(\overset{2}{\mathfrak{t}}u(x))$. Note that the fact that Lagrangian can be just a function on $\overset{2}{\mathbb{T}}M$ is due to the existence of the canonical volume form $dx^1 \wedge dx^2$ on \mathbb{R}^2 . We can therefore identify scalar densities, i.e. objects that can be integrated, with functions.

Variations of u are maps δu from D to $\mathbb{T}M$ covering u :

$$\begin{array}{ccc} D & \xrightarrow{\delta u} & \mathbb{T}M \\ & \searrow u & \downarrow \tau_M \\ & & M \end{array}$$

and coming from homotopies $\chi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^2, M)$ of maps $\mathcal{C}^\infty(\mathbb{R}^2, M)$. If $u(x) = \chi(0, x)$, then $\delta u(x)$ is a vector tangent to the curve $t \mapsto \chi(t, x)$ at $t = 0$. In the following we perform the standard calculus of a variation of S with respect to the variation δu :

$$\begin{aligned} \langle dS, \delta u \rangle &= \frac{d}{dt} \Big|_{t=0} \int_D L(\chi^a(t, x), \frac{\partial}{\partial x^1} \chi^b(t, x), \frac{\partial}{\partial x^2} \chi^c(t, x)) dx^1 dx^2 = \\ &= \int_D \left(\frac{\partial L}{\partial q^a} \delta u^a + \frac{\partial L}{\partial \dot{u}_1^b} \delta \dot{u}_1^b + \frac{\partial L}{\partial \dot{u}_2^c} \delta \dot{u}_2^c \right) dx^1 dx^2 = \dots \end{aligned}$$

where

$$\delta \dot{u}_i^a(x) = \frac{\partial \chi^a}{\partial t \partial x_i}(0, x).$$

Using the Stokes theorem, we obtain

$$\dots = \int_D \left(\frac{\partial L}{\partial q^a} - \frac{\partial}{\partial x_1} \frac{\partial L}{\partial \dot{q}_1^a} - \frac{\partial}{\partial x_2} \frac{\partial L}{\partial \dot{q}_2^a} \right) \delta u^a dx^1 dx^2 + \int_{\partial D} \left(\frac{\partial L}{\partial \dot{q}_1^a} dx_2 - \frac{\partial L}{\partial \dot{q}_2^a} dx_1 \right) \delta u^a,$$

where the last integral is calculated over ∂D oriented as in the Stokes theorem, using the canonical orientation of \mathbb{R}^2 . Looking for the stationary points of the action functional S we put the condition $\langle dS, \delta u \rangle = 0$ for every δu , which means that

$$(3.1) \quad \frac{\partial L}{\partial q^a} - \frac{\partial}{\partial x_1} \left(\frac{\partial L}{\partial \dot{q}_1^a} \right) - \frac{\partial}{\partial x_2} \left(\frac{\partial L}{\partial \dot{q}_2^a} \right) = 0$$

on the disc and

$$\frac{\partial L}{\partial q_1^a} dx_2 - \frac{\partial L}{\partial q_2^a} dx_1 = 0$$

on the boundary ∂D . The equation (3.1) is traditionally called the Euler-Lagrange equation. The boundary term is an analog of the momentum in the Classical Mechanics. The momentum evaluated on a variation δu gives a one-form on \mathbb{R}^2 (to be integrated over ∂D). It follows that the phase space is a space of covectors on M with values in the cotangent space of \mathbb{R}^2 . The cotangent bundle of \mathbb{R}^2 is trivial, with the fiber being just $(\mathbb{R}^2)^* \simeq \mathbb{R}^2$, but we will keep the notation $(\mathbb{R}^2)^*$ and use the basis $(dx^2, -dx^1)$ to identify it with \mathbb{R}^2 . The phase space can be therefore identified with

$$\mathsf{T}^*M \otimes_M (\mathbb{R}^2)^* \simeq \mathsf{T}^*M \otimes_M \mathbb{R}^2 \simeq \mathsf{T}^*M \times_M \mathsf{T}^*M = \overset{2}{\mathsf{T}}^*M.$$

The Legendre map which associates a momentum to an infinitesimal configuration will be discussed later in the section.

In the calculation of the differential of the action functional we have used implicitly a mapping

$$(3.2) \quad \kappa : \overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}} M \rightarrow \overset{2}{\mathsf{T}} \mathsf{T} M$$

defined as follows. Starting from a homotopy χ we can construct an element of $\overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}} M$ by taking the tangent vector of the curve $\mathbb{R} \ni t \mapsto \overset{2}{\mathsf{T}} \chi(t, x) \in \overset{2}{\mathsf{T}} M$ at $t = 0$. From the same homotopy we get $\mathbb{R}^2 \ni x \mapsto \mathsf{t} \chi(t, x) \in \mathsf{T} M$. The first jet at x of the last map is an element of $\overset{2}{\mathsf{T}} \mathsf{T} M$. Therefore

$$(3.3) \quad \kappa(\mathsf{t} \overset{2}{\mathsf{T}} \chi(0, x)) = \overset{2}{\mathsf{T}} \mathsf{t} \chi(0, x).$$

The above definition is analogous to the definition of the canonical flip $\kappa_M : \mathsf{T} \mathsf{T} M \rightarrow \mathsf{T} \mathsf{T} M$.

Using the local coordinate system (q^a) on M , we can construct local coordinates on $\overset{2}{\mathsf{T}} \mathsf{T} M$ and $\overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}} M$. The coordinates on $\overset{2}{\mathsf{T}} \mathsf{T} M$ will be denoted by

$$(3.4) \quad (q^a, \delta q^b, \dot{q}_1^c, \delta \dot{q}_1^d, \dot{q}_2^e, \delta \dot{q}_2^f)$$

and the ones on $\overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}} M$ by

$$(3.5) \quad (q^a, \dot{q}_1^b, \dot{q}_2^c, \delta \dot{q}^d, \delta \dot{q}_1^e, \delta \dot{q}_2^f).$$

Since

$$\kappa(q^a, \dot{q}_1^b, \dot{q}_2^c, \delta \dot{q}^d, \delta \dot{q}_1^e, \delta \dot{q}_2^f) = (q^a, \delta q^b, \dot{q}_1^c, \delta \dot{q}_1^d, \dot{q}_2^e, \delta \dot{q}_2^f),$$

using the same notation for coordinates in different spaces does not lead to any confusion.

4 The Lagrangian side

In the previous section we recognized the phase space as $\overset{2}{\mathsf{T}}^*M$. In the following it will be useful to remember that the space $\overset{2}{\mathsf{T}}^*M$ has been identified with $\mathsf{T}^*M \otimes_M (\mathbb{R}^2)^*$. We shall define now a pairing between the space of jets of maps $p \in \mathcal{C}^\infty(\mathbb{R}^2, \overset{2}{\mathsf{T}}^*M)$ at the point $(0, 0)$, i.e. $\overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}}^*M$, and jets of variations $\delta u \in \mathcal{C}^\infty(\mathbb{R}^2, \mathsf{T} M)$ at $(0, 0)$, i.e. $\overset{2}{\mathsf{T}} \mathsf{T} M$. (Using the structure of \mathbb{R}^2 we identify $\mathsf{J}^1(\mathbb{R}^2, \overset{2}{\mathsf{T}}^*M)$ with $\mathbb{R}^2 \times \overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}}^*M$ and $\mathsf{J}^1(\mathbb{R}^2, \mathsf{T} M)$ with $\mathbb{R}^2 \times \overset{2}{\mathsf{T}} \mathsf{T} M$, like in the case of jets of maps $\mathcal{C}^\infty(\mathbb{R}^2, M)$).

The space $\overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}}^*M$ has two vector bundle structures: the canonical one over $\overset{2}{\mathsf{T}}^*M$, and the tangent one over $\overset{2}{\mathsf{T}} M$. The tangent projection $\overset{2}{\mathsf{T}} \pi_M^2$ can be constructed as follows: an element $w \in \overset{2}{\mathsf{T}} \overset{2}{\mathsf{T}}^*M$ has a representative $\eta : \mathbb{R}^2 \rightarrow \overset{2}{\mathsf{T}}^*M$, $w = \mathsf{t} \eta(0, 0)$. The projection $\overset{2}{\mathsf{T}} \pi_M^2$ is given by

$$\overset{2}{\mathsf{T}} \pi_M^2(w) = \mathsf{t}(\pi_M^2 \circ \eta)(0, 0).$$

Both bundles are vector bundles which form together a *double vector bundle* [16]:

$$(4.1) \quad \begin{array}{ccc} & {}^2\mathbb{T}^*M & \\ \tau_{\mathbb{T}^*M}^2 \swarrow & & \searrow \tau_{\mathbb{T}M}^2 \\ {}^2\mathbb{T}^*M & & {}^2\mathbb{T}M \\ \pi_M^2 \searrow & & \swarrow \tau_M^2 \\ & M & \end{array}$$

Since the space of infinitesimal configurations ${}^2\mathbb{T}M$ has also a vector bundle structure over M , the cotangent bundle $\mathbb{T}^*{}^2\mathbb{T}M$ is a double vector bundle with the canonical projection on ${}^2\mathbb{T}M$ and the second projection on ${}^2\mathbb{T}^*M$ (cf. for example [10, 7]). The second projection is defined as follows: any element a of $\mathbb{T}_v^*{}^2\mathbb{T}M$ is a linear function on the vector space $\mathbb{T}_v{}^2\mathbb{T}M$. Restricting a to the subspace of vectors tangent to the fibre of projection τ_M^2 (which is isomorphic to the fibre itself), we can associate with it an element of ${}^2\mathbb{T}^*M$. The projection will be denoted by ξ . The structure of the double vector bundle $\mathbb{T}^*{}^2\mathbb{T}M$ can be put into the following diagram:

$$(4.2) \quad \begin{array}{ccc} & \mathbb{T}^*{}^2\mathbb{T}M & \\ \xi \swarrow & & \searrow \pi_{\mathbb{T}^2M} \\ {}^2\mathbb{T}^*M & & {}^2\mathbb{T}M \\ \pi_M^2 \searrow & & \swarrow \tau_M^2 \\ & M & \end{array}$$

The double vector bundle $\mathbb{T}^*{}^2\mathbb{T}M$ is canonically isomorphic with $\mathbb{T}^*{}^2\mathbb{T}^*M$.

Let ${}^2\mathfrak{t}p$ and ${}^2\mathfrak{t}\delta u$ be elements of ${}^2\mathbb{T}^*{}^2\mathbb{T}^*M$ and ${}^2\mathbb{T}^*{}^2\mathbb{T}M$, respectively, such that they have the same projection on ${}^2\mathbb{T}M$. Let p and δu denote the representatives covering the same map $u : \mathbb{R}^2 \rightarrow M$. Interpreting an element of ${}^2\mathbb{T}^*M$ as a covector on M with values in $(\mathbb{R}^2)^*$, we can define the mapping

$$\mathbb{R}^2 \ni (x^1, x^2) \mapsto \langle p(x^1, x^2), \delta u(x^1, x^2) \rangle \in (\mathbb{R}^2)^*,$$

where the target space is the fiber of $\mathbb{T}^*\mathbb{R}^2$. The mapping can be viewed as a one-form on \mathbb{R}^2 . The differential of the above one-form is a two-form on \mathbb{R}^2 which, due to the existence of the canonical form $dx^1 \wedge dx^2$, can be identified with a function. The formula

$$(4.3) \quad \langle \langle {}^2\mathfrak{t}p, {}^2\mathfrak{t}\delta u \rangle \rangle dx^1 \wedge dx^2 = d\langle p, \delta u \rangle(0, 0)$$

defines a pairing between ${}^2\mathbb{T}^*{}^2\mathbb{T}^*M$ and ${}^2\mathbb{T}^*{}^2\mathbb{T}M$ over ${}^2\mathbb{T}M$. The above pairing is of course degenerate.

Using the basis $(dx^2, -dx^1)$ of sections of $\mathbb{T}^*\mathbb{R}^2$ and the local coordinate system (q^a) on M , we can construct local linear coordinates (q^a, p_b^1, p_c^2) in the phase space ${}^2\mathbb{T}^*M$, associated with local sections $(dq^b \otimes dx^2, -dq^c \otimes dx^1)$. Note that in our convention the coordinates (q^a, p_b^1, p_c^2) are associated with the element $p_b^1 dq^b \otimes dx^2 - p_c^2 dq^c \otimes dx^1$. In the space of jets ${}^2\mathbb{T}^*{}^2\mathbb{T}^*M$ we have therefore the adapted system of coordinates

$$(q^a, p_b^1, p_c^2, \dot{q}_1^d, \dot{p}_{e1}^1, \dot{p}_{f1}^2, \dot{q}_2^g, \dot{p}_{h2}^1, \dot{p}_{k2}^2).$$

On the other hand, in the space ${}^2\mathbb{T}M$ of jets of variations we have also an adapted set of coordinates induced by the coordinates on M :

$$(q^a, \delta q^b, \dot{q}_1^c, \delta \dot{q}_1^d, \dot{q}_2^e, \delta \dot{q}_2^f).$$

In the above coordinates the evaluation reads:

$$(4.4) \quad \langle\langle \mathfrak{t}p, \mathfrak{t}\delta u \rangle\rangle = (\dot{p}_{a1}^1 + \dot{p}_{a2}^2)\delta q^a + p_b^2\delta\dot{q}_2^b + p_c^1\delta\dot{q}_1^c.$$

Now we are ready to define the mapping

$$\alpha : \overset{2}{\mathbb{T}}\overset{2}{\mathbb{T}}^*M \longrightarrow \mathbb{T}^*\overset{2}{\mathbb{T}}M$$

by the condition

$$\langle\alpha(w), v\rangle = \langle\langle w, \kappa(v) \rangle\rangle,$$

where $w \in \overset{2}{\mathbb{T}}\overset{2}{\mathbb{T}}^*M$, $v \in \mathbb{T}^*\overset{2}{\mathbb{T}}M$, and the evaluation on the left side of the equation is the canonical evaluation between $\overset{2}{\mathbb{T}}\overset{2}{\mathbb{T}}^*M$ and $\mathbb{T}^*\overset{2}{\mathbb{T}}M$. The evaluation on the right side is the one defined in (4.4). In local coordinates the mapping α reads

$$\alpha(q^a, p_b^1, p_c^2, \dot{q}_1^d, \dot{p}_{e1}^1, \dot{p}_{f1}^2, \dot{q}_2^g, \dot{p}_{h2}^1, \dot{p}_{k2}^2) = (q^a, \dot{q}_1^d, \dot{q}_2^g, \dot{p}_{f1}^1 + \dot{p}_{f2}^2, p_b^1, p_c^2).$$

The mapping α is an analog of $\alpha_M : \mathbb{T}\mathbb{T}^*M \rightarrow \mathbb{T}^*\mathbb{T}M$ used by Tulczyjew in the autonomous mechanics.

Definition 4.1. The *phase equations* for the system with the Lagrangian L are induced by the subset

$$(4.5) \quad \mathcal{D} = \alpha^{-1}(\mathrm{d}L(\overset{2}{\mathbb{T}}M))$$

in an obvious way: a mapping $p : \mathbb{R}^2 \supset \mathcal{O} \rightarrow \overset{2}{\mathbb{T}}^*M$, where \mathcal{O} is an open subset of \mathbb{R}^2 , is a solution of the phase equations if

$$\alpha(\mathfrak{t}p(x, y)) = \mathrm{d}L(\mathfrak{t}(\pi_M^2 \circ p)(x, y))$$

for any $(x, y) \in \mathcal{O}$.

The important difference with the case of Classical Mechanics is that α is no longer an isomorphism, therefore α^{-1} is a relation only, not a mapping. In local coordinates we obtain

$$(4.6) \quad \dot{p}_{a1}^1 + \dot{p}_{a2}^2 = \frac{\partial L}{\partial q^a}, \quad p_b^1 = \frac{\partial L}{\partial \dot{q}_1^b}, \quad p_c^2 = \frac{\partial L}{\partial \dot{q}_2^c}.$$

The *Legendre map*, that associates a momentum to an infinitesimal configuration, is defined as:

$$(4.7) \quad \lambda_L : \overset{2}{\mathbb{T}}M \longrightarrow \overset{2}{\mathbb{T}}^*M, \quad \lambda_L = \mathrm{d}L \circ \xi.$$

In coordinates it reads

$$\lambda_L(q^a, \dot{q}_1^b, \dot{q}_2^c) = \left(q^a, \frac{\partial L}{\partial \dot{q}_1^b}, \frac{\partial L}{\partial \dot{q}_2^c} \right)$$

The *Euler-Lagrange equations* for configurations

$$u : \mathbb{R}^2 \ni (x^1, x^2) \longmapsto (q^a(x^1, x^2)) \in M$$

can be formulated in the following way

$$(4.8) \quad \mathfrak{t}\lambda_L = \alpha^{-1} \circ \mathrm{d}L$$

that in coordinates reads

$$\begin{aligned} \dot{q}_1^a &= \frac{\partial q^a}{\partial x^1}, \\ \dot{q}_2^a &= \frac{\partial q^a}{\partial x^2}, \\ \frac{\partial L}{\partial q^a} - \frac{\partial}{\partial x^1} \frac{\partial L}{\partial \dot{q}_1^a} - \frac{\partial}{\partial x^2} \frac{\partial L}{\partial \dot{q}_2^a} &= 0. \end{aligned}$$

The equations we obtained are in full agreement with equations commonly accepted in Classical Field Theory the theory (cf. [3, 13]).

All the structure needed for generating the phase equations from the Lagrangian can be presented in the following diagram:

$$(4.9) \quad \begin{array}{ccccc} & {}^2\mathbb{T}^2\mathbb{T}^*M & \xrightarrow{\alpha} & \mathbb{T}^*\mathbb{T}^2M & \\ \tau_{\mathbb{T}^2}^2 \swarrow & & & \searrow \xi & \\ {}^2\mathbb{T}^*M & \xrightarrow{id} & {}^2\mathbb{T}^*M & & \xrightarrow{\pi_{\mathbb{T}^2}M} \\ \pi_M^2 \searrow & & \searrow \pi_M^2 & & \\ & {}^2\mathbb{T}M & \xrightarrow{id} & {}^2\mathbb{T}M & \\ \tau_M^2 \swarrow & & \swarrow \tau_M^2 & & \\ & M & \xrightarrow{id} & M & \end{array}$$

5 The Hamiltonian side

In the autonomous mechanics the basic structure is the canonical symplectic form ω_M on the cotangent bundle being the phase space. Using the form ω_M , we associate the Hamiltonian vector field to any Hamiltonian – a smooth function on the phase space. In our case, the phase space is not a symplectic manifold any more, but still we can establish a correspondence between the cotangent bundle of the phase space and the space ${}^2\mathbb{T}^2\mathbb{T}^*M$ of jets of maps from $\mathcal{C}^\infty(\mathbb{R}^2, {}^2\mathbb{T}^*M)$. We present here two ways of constructing the appropriate mapping.

The first method is based on the fact that the phase bundle is a vector bundle over M , so we have the canonical antisymplectomorphism (cf. [7, 16])

$$\mathbb{T}^*{}^2\mathbb{T}^*M \simeq \mathbb{T}^*{}^2\mathbb{T}M.$$

Denoting the above antisymplectomorphism by \mathcal{R} and composing it with α we obtain

$$(5.1) \quad \beta = \alpha \circ \mathcal{R} : {}^2\mathbb{T}^2\mathbb{T}^*M \longrightarrow \mathbb{T}^*{}^2\mathbb{T}^*M$$

Since \mathcal{R} and α are double vector bundle morphisms, we obtain the following diagram for the mapping β :

$$(5.2) \quad \begin{array}{ccccc} & \mathbb{T}^*{}^2\mathbb{T}^*M & \xleftarrow{\beta} & {}^2\mathbb{T}^2\mathbb{T}^*M & \\ \pi_{\mathbb{T}^2}^2 \swarrow & & & \searrow \tau_{\mathbb{T}^2}^2 & \\ {}^2\mathbb{T}^*M & \xleftarrow{id} & {}^2\mathbb{T}^*M & & \xleftarrow{{}^2\mathbb{T}\pi_M^2} \\ \pi_M^2 \searrow & & \searrow \pi_M^2 & & \\ & {}^2\mathbb{T}M & \xleftarrow{id} & {}^2\mathbb{T}M & \\ \tau_M^2 \swarrow & & \swarrow \tau_M^2 & & \\ & M & \xleftarrow{id} & M & \end{array}$$

The projection $\zeta : \mathbb{T}^*{}^2\mathbb{T}^*M \rightarrow {}^2\mathbb{T}M$ is an analog of ξ in (4.2) and a particular instance of the canonical projection $\mathbb{T}^*E^* \rightarrow E$ for a vector bundle E . If we use the system of local coordinates on ${}^2\mathbb{T}^2\mathbb{T}^*M$ as previously,

$$(q^a, p_b^1, p_c^2, \dot{q}_1^d, \dot{p}_{e1}^1, \dot{p}_{f1}^2, \dot{q}_2^g, \dot{p}_{h2}^1, \dot{p}_{k2}^2),$$

and the system of coordinates

$$(q^a, p_b^1, p_c^2, \varphi_d, \psi_1^e, \psi_2^f),$$

derived from the coordinates (q^a, p_b^1, p_c^2) and associated to the local sections (dq^a, dp_b^1, dp_c^2) , we get that

$$(5.3) \quad \beta(q^a, p_b^1, p_c^2, \dot{q}_1^d, \dot{p}_{e1}^1, \dot{p}_{f1}^2, \dot{q}_2^g, \dot{p}_{h2}^1, \dot{p}_{k2}^2) = (q^a, p_b^1, p_c^2, -\dot{p}_{d1}^1 - \dot{p}_{d2}^2, \dot{q}_1^e, \dot{q}_2^f).$$

For any Hamiltonian

$$H : \overset{2}{\mathbb{T}}^* M \longrightarrow \mathbb{R},$$

the phase dynamic is represented by the subset

$$(5.4) \quad \mathcal{D} = \beta^{-1}(\mathrm{d}H(\overset{2}{\mathbb{T}}^* M)) \subset \overset{2}{\mathbb{T}}\overset{2}{\mathbb{T}}^* M.$$

Also in this case, β^{-1} is a relation only. In local coordinates we obtain the phase equations

$$\begin{aligned} -\dot{p}_{d1}^1 - \dot{p}_{d2}^2 &= \frac{\partial H}{\partial q^d}, \\ \dot{q}_1^e &= \frac{\partial H}{\partial p_e^1}, \\ \dot{q}_2^f &= \frac{\partial H}{\partial p_f^2}. \end{aligned}$$

An alternative way of constructing the mapping β does not refer to the map α . Let us denote by pr_1, pr_2 the projections on the first and the second factor of $\overset{2}{\mathbb{T}}^* M = \mathbb{T}M \times_M \mathbb{T}M$. In local coordinates, we have

$$\begin{aligned} pr_1 : \overset{2}{\mathbb{T}}^* M \ni (q^a, p_b^1, p_c^2) &\longmapsto (q^a, p_b^1) \in \mathbb{T}^* M, \\ pr_2 : \overset{2}{\mathbb{T}}^* M \ni (q^a, p_b^1, p_c^2) &\longmapsto (q^a, p_b^2) \in \mathbb{T}^* M. \end{aligned}$$

Applying the tangent lift to the both projections we obtain

$$\begin{aligned} \mathbb{T}pr_1 : \mathbb{T}\overset{2}{\mathbb{T}}^* M \ni (q^a, p_b^1, p_c^2, \dot{q}^a, \dot{p}_b^1, \dot{p}_c^2) &\longmapsto (q^a, p_b^1, \dot{q}^a, \dot{p}_b^1) \in \mathbb{T}\mathbb{T}^* M, \\ \mathbb{T}pr_2 : \mathbb{T}\overset{2}{\mathbb{T}}^* M \ni (q^a, p_b^1, p_c^2, \dot{q}^a, \dot{p}_b^1, \dot{p}_c^2) &\longmapsto (q^a, p_b^2, \dot{q}^a, \dot{p}_b^2) \in \mathbb{T}\mathbb{T}^* M. \end{aligned}$$

Composing the cartesian product of the above tangent mappings with the inclusion

$$\iota : \overset{2}{\mathbb{T}}\overset{2}{\mathbb{T}}^* M \hookrightarrow \mathbb{T}\overset{2}{\mathbb{T}}^* M \times \mathbb{T}\overset{2}{\mathbb{T}}^* M,$$

we get

$$\begin{aligned} (\mathbb{T}pr_1 \times \mathbb{T}pr_2) \circ \iota : \overset{2}{\mathbb{T}}\overset{2}{\mathbb{T}}^* M &\longrightarrow \mathbb{T}\mathbb{T}^* M \times \mathbb{T}\mathbb{T}^* M \\ (q^a, p_b^1, p_c^2, \dot{q}_1^d, \dot{p}_{e1}^1, \dot{p}_{f1}^2, \dot{q}_2^g, \dot{p}_{h2}^1, \dot{p}_{k2}^2) &\longmapsto ((q^a, p_b^1, \dot{q}_1^d, \dot{p}_{1b}^1), (q^a, p_b^2, \dot{q}_2^g, \dot{p}_{2b}^2)). \end{aligned}$$

To the both factors of the image of the composition $(\mathbb{T}pr_1 \times \mathbb{T}pr_2) \circ \iota$ we apply the canonical map $\beta_M : \mathbb{T}\mathbb{T}^* M \rightarrow \mathbb{T}^*\mathbb{T}^* M$ that comes from the canonical symplectic form ω_M on the cotangent bundle $\mathbb{T}^* M$. The target space of the composition

$$(\beta_M \times \beta_M) \circ (\mathbb{T}pr_1 \times \mathbb{T}pr_2) \circ \iota$$

is therefore

$$\mathbb{T}^*\mathbb{T}^* M \times \mathbb{T}^*\mathbb{T}^* M$$

which, in turn, can be mapped to $\mathbb{T}^* \overset{2}{\mathbb{T}}^* M$ by means of the phase lift of the inclusion

$$j : \overset{2}{\mathbb{T}}^* M \hookrightarrow \mathbb{T}^* M \times \mathbb{T}^* M.$$

Finally, we end up with the map

$$(5.5) \quad j^* \circ (\beta_M \times \beta_M) \circ (\mathbb{T}pr_1 \times \mathbb{T}pr_2) \circ \iota : \overset{2}{\mathbb{T}} \overset{2}{\mathbb{T}}^* M \longrightarrow \mathbb{T}^* \overset{2}{\mathbb{T}}^* M.$$

Proposition 5.1. *The mappings defined in (5.1) and (5.5) coincide, i.e.*

$$\beta = j^* \circ (\beta_M \times \beta_M) \circ (\mathbb{T}pr_1 \times \mathbb{T}pr_2) \circ \iota.$$

Proof: Let us start with recalling the definition of the canonical isomorphism \mathcal{R}_E for a general vector bundle $E \rightarrow M$. The graph of \mathcal{R}_E is the Lagrangian submanifold generated in $\mathbb{T}^*(E \times E^*) \simeq \mathbb{T}^*E \times \mathbb{T}^*E^*$ by the evaluation function

$$E \times_M E^* \ni (y, a) \longmapsto \langle a, y \rangle \in \mathbb{R},$$

defined on the submanifold $E \times_M E^*$ of $E \times E^*$. We see that, by definition, for any element $\varphi \in \mathbb{T}^*E$, its image $\mathcal{R}_E(\varphi)$ has the same projections onto E and E^* as φ . If we take now two curves

$$\begin{aligned} \mathbb{R} \ni t &\longmapsto \gamma(t) \in E, \\ \mathbb{R} \ni t &\longmapsto \eta(t) \in E^*, \end{aligned}$$

covering the same curve in M and such that $\gamma(0)$ and $\eta(0)$ are equal to the projections of φ to E and E^* respectively, we can write

$$\langle \langle \varphi, \mathcal{R}_E(\varphi) \rangle, (\mathfrak{t}\gamma(0), \mathfrak{t}\eta(0)) \rangle = \frac{d}{dt} \Big|_{t=0} \langle \eta(t), \gamma(t) \rangle,$$

or

$$(5.6) \quad \langle \mathcal{R}_E(\varphi), \mathfrak{t}\eta(0) \rangle = \frac{d}{dt} \Big|_{t=0} \langle \eta(t), \gamma(t) \rangle - \langle \varphi, \mathfrak{t}\gamma(0) \rangle.$$

Let now $\psi : \mathbb{R}^2 \rightarrow \mathbb{T}^*M$ be a homotopy such that $\psi(0, 0)$ is the projection of v and w on \mathbb{T}^*M , the curve $a \rightarrow \psi(a, 0)$ is a representative of v , and $b \rightarrow \psi(0, b)$ is a representative of w . Using the definitions of ω_M and ϑ_M (see (1.5, 1.6)), we can write that

$$\begin{aligned} \omega_M(v, w) &= \frac{d}{db} \Big|_{b=0} \theta_M(\mathfrak{t}\psi(\cdot, b)(0)) - \frac{d}{da} \Big|_{a=0} \theta_M(\mathfrak{t}\psi(a, \cdot)(0)) = \\ &= \frac{d}{db} \Big|_{b=0} \langle \psi(0, b), \mathfrak{t}(\pi_M \circ \psi(\cdot, b))(0) \rangle - \frac{d}{da} \Big|_{a=0} \langle \psi(a, 0), \mathfrak{t}(\pi_M \circ \psi(a, \cdot))(0) \rangle. \end{aligned}$$

We can simplify the above formula a little bit introducing curves

$$p_1(a) = \psi(a, 0), \quad p_2(b) = \psi(0, b),$$

which represent v and w , respectively, and a homotopy in M defined by

$$\rho = \pi_M \circ \psi.$$

In the new notation we have

$$(5.7) \quad \omega_M(v, w) = \frac{d}{db} \Big|_{t=0} \langle p_2(t), \mathfrak{t}\rho(\cdot, b)(0) \rangle - \frac{d}{da} \Big|_{a=0} \langle p_1(a), \mathfrak{t}(\rho(a, \cdot))(0) \rangle.$$

Now, we can start the main part of the proof, which is done by simple calculations. We are going to show that the following diagram is commutative:

$$(5.8) \quad \begin{array}{ccc} & \overset{2}{\mathbb{T}} \overset{2}{\mathbb{T}}^* M & \\ \tilde{\beta} \swarrow & & \searrow \alpha \\ \mathbb{T}^* \overset{2}{\mathbb{T}}^* M & \xleftarrow{\mathcal{R}} & \mathbb{T}^* \overset{2}{\mathbb{T}} M \end{array}$$

We have used the symbol $\tilde{\beta}$ for the long expression $j^* \circ (\beta_M \times \beta_M) \circ (\mathbb{T}pr_1 \times \mathbb{T}pr_1) \circ \iota$ and the shorter \mathcal{R} for $\mathcal{R}_{\mathbb{T}^2 M}$. The commutativity of the diagram (5.8) means that, for any $w \in \overset{2}{\mathbb{T}} \overset{2}{\mathbb{T}}^* M$ and $u \in \mathbb{T}^* \overset{2}{\mathbb{T}}^* M$ such that they have the same projection on $\overset{2}{\mathbb{T}}^* M$, we have

$$(5.9) \quad \langle \tilde{\beta}(w), u \rangle = \langle \mathcal{R} \circ \alpha(w), u \rangle.$$

To calculate the right-hand side of the equation (5.9), we need the representatives of u and w . Let

$$\mathbb{R} \ni t \longmapsto \eta(t) = (\eta_1(t), \eta_2(t)) \in \overset{2}{\mathbb{T}}^* M$$

be a curve that represents the vector u , i.e.

$$u = \mathfrak{t}\eta(0).$$

The element w of $\overset{2}{\mathbb{T}} \overset{2}{\mathbb{T}}^* M$ is represented by the mapping

$$\mathbb{R}^2 \ni (x, y) \longmapsto p(x, y) = (p_1(x, y), p_2(x, y)) \in \overset{2}{\mathbb{T}}^* M.$$

We can also choose a mapping

$$\mathbb{R}^3 \ni (t, x, y) \longmapsto \chi(t, x, y) \in M$$

such that

$$\chi(t, 0, 0) = \pi_M^2 \circ \eta(t), \chi(0, x, y) = \pi_M^2 \circ p(x, y).$$

Let us start with the R.H.S. of (5.9) using (5.6):

$$\begin{aligned} \langle \mathcal{R} \circ \alpha(w), u \rangle &= \langle \mathcal{R} \circ \alpha(\mathfrak{t}p(0, 0)), \mathfrak{t}\eta(0) \rangle = \\ &= \frac{d}{dt} \Big|_{t=0} \langle \eta(t), \mathfrak{t}\chi(t, \cdot, \cdot)(0, 0) \rangle - \langle \alpha(\mathfrak{t}p(0, 0)), \mathfrak{t}\mathfrak{t}\chi(\cdot, \cdot, \cdot)(0, 0, 0) \rangle. \end{aligned}$$

Using the definition of α and the tangent evaluation, we can write the above expression as

$$\frac{d}{dt} \Big|_{t=0} \langle \eta(t), \mathfrak{t}\chi(t, \cdot, \cdot)(0, 0) \rangle - \langle \mathfrak{t}p(0, 0), \mathfrak{t}\mathfrak{t}\chi(\cdot, \cdot, \cdot)(0, 0, 0) \rangle.$$

Finally, replacing η by the pair (η_1, η_2) and using the definition of the evaluation $\langle \cdot, \cdot \rangle$, we obtain the final form

$$(5.10) \quad \begin{aligned} \langle \mathcal{R} \circ \alpha(w), u \rangle &= \frac{d}{dt} \Big|_{t=0} \langle \eta_1(t), \mathfrak{t}\chi(t, \cdot, 0)(0) \rangle + \frac{d}{dt} \Big|_{t=0} \langle \eta_2(t), \mathfrak{t}\chi(t, 0, \cdot)(0) \rangle + \\ &= - \frac{d}{dx} \Big|_{x=0} \langle p_1(x, 0), \mathfrak{t}\chi(\cdot, x, 0)(0) \rangle - \frac{d}{dy} \Big|_{y=0} \langle p_2(0, y), \mathfrak{t}\chi(\cdot, 0, y)(0) \rangle. \end{aligned}$$

Before we start with the L.H.S., let us note that (using representatives) we have

$$\mathfrak{t}(\mathfrak{t}p(0, 0)) = (\mathfrak{t}p_1(\cdot, 0)(0), \mathfrak{t}p_2(\cdot, 0)(0), \mathfrak{t}p_1(0, \cdot)(0), \mathfrak{t}p_2(0, \cdot)(0))$$

and

$$\mathbb{T}pr_1 \times \mathbb{T}pr_2(\mathfrak{t}p_1(\cdot, 0)(0), \mathfrak{t}p_2(\cdot, 0)(0), \mathfrak{t}p_1(0, \cdot)(0), \mathfrak{t}p_2(0, \cdot)(0)) = (\mathfrak{t}p_1(\cdot, 0)(0), \mathfrak{t}p_2(0, \cdot)(0)).$$

Now, we can do our calculation as follows:

$$(5.11) \quad \langle j^* \circ (\beta_M \times \beta_M) \circ (\mathbb{T}pr_1 \times \mathbb{T}pr_1) \circ \iota(w), u, \rangle = \\ \langle j^* \circ (\beta_M \times \beta_M) \circ (\mathbb{T}pr_1 \times \mathbb{T}pr_1) \circ \iota(\mathfrak{t}p(0, 0)), \mathfrak{t}\eta(0), \rangle = \\ \langle (\beta_M \times \beta_M)(\mathfrak{t}p_1(\cdot, 0)(0), \mathfrak{t}p_2(0, \cdot)(0)), \mathbb{T}j(\mathfrak{t}\eta(0)) \rangle.$$

Taking into account that

$$\mathbb{T}j(\mathfrak{t}\eta(0)) = (\mathfrak{t}\eta_1(0), \mathfrak{t}\eta_2(0)),$$

we can express (5.11) as

$$\langle \beta_M(\mathfrak{t}p_1(\cdot, 0)(0)), \mathfrak{t}\eta_1(0) \rangle + \langle \beta_M(\mathfrak{t}p_2(0, \cdot)(0)), \mathfrak{t}\eta_2(0) \rangle = \\ \omega_M(\mathfrak{t}p_1(\cdot, 0)(0), \mathfrak{t}\eta_1(0)) + \omega_M(\mathfrak{t}p_2(0, \cdot)(0), \mathfrak{t}\eta_2(0)).$$

Now, we can use (5.7) for the final form of the L.H.S of equation (5.9):

$$(5.12) \quad \langle \tilde{\beta}(w), u \rangle = \frac{d}{db|_{b=0}} \langle \eta_1(b), \mathfrak{t}\chi(b, \cdot, 0)(0) \rangle - \frac{d}{da|_{a=0}} \langle p_1(a, 0), \mathfrak{t}\chi(\cdot, a, 0)(0) \rangle \\ \frac{d}{db|_{b=0}} \langle \eta_2(b), \mathfrak{t}\chi(b, 0, \cdot)(0) \rangle - \frac{d}{da|_{a=0}} \langle p_2(0, a), \mathfrak{t}\chi(\cdot, 0, a)(0) \rangle.$$

It is easy to see that (5.10) and (5.12) coincide. \square

The above proof is very technical, but tracing the calculations one can make at least one important observation. In the final form of the R.H.S. and the L.H.S. one can see that momentum (p_1, p_2) is always evaluated on $\mathfrak{t}\chi(\cdot, x, y)(0)$, i.e. on a tangent vector with respect to the first parameter t of χ . The latter can be denoted by δu and understood as a variation of the configuration u . In (5.10) it appears by definition from the tangent evaluation, but in (5.12) it comes from the calculation. The geometrical structure reflects therefore the idea that the momentum is to be evaluated on the variation rather than on the infinitesimal configuration. In Classical Mechanics the difference is not visible, because both, infinitesimal configurations and variations, are tangent vectors. In our case the difference is visible but does not have much consequence, because the bundle of momenta is the dual vector bundle of the bundle of infinitesimal configurations. In more general cases of Field Theory it is no longer true. The Hamiltonian side is then more complicated.

In Classical Mechanics the mapping β_M comes from the canonical symplectic form on the cotangent bundle \mathbb{T}^*M . We can therefore ask, whether the mapping β we have just constructed is related to some tensor field, which can be regarded as a canonical structure on the phase space. It is easy to see that, indeed, the mapping β is related to the field

$$\overset{2}{\omega}_M \in \text{Sec}(\overset{2}{\mathbb{T}}^* \overset{2}{\mathbb{T}}^*M \otimes \mathbb{T}^* \overset{2}{\mathbb{T}}^*M),$$

which in local coordinates reads

$$\overset{2}{\omega}_M = d^1q^a \otimes dp^1_a + d^2q^b \otimes dp^2_b - d^1p^1_c \otimes dq^c - d^2p^2_d \otimes dq^d.$$

Here, $(d^i q^a, d^i p^1_b, d^i p^1_b)$ is the basis in the first (if $i = 1$) or the second (if $i = 2$) factor of $\overset{2}{\mathbb{T}}^* \overset{2}{\mathbb{T}}^*M = \mathbb{T}^* \overset{2}{\mathbb{T}}^*M \times_{\mathbb{T}^{2*}M} \mathbb{T}^* \overset{2}{\mathbb{T}}^*M$. Since

$$\overset{2}{\mathbb{T}}^* \overset{2}{\mathbb{T}}^*M \otimes \mathbb{T}^* \overset{2}{\mathbb{T}}^*M \simeq \mathbb{T}^* \overset{2}{\mathbb{T}}^*M \otimes \mathbb{T}^* \overset{2}{\mathbb{T}}^*M \times_{\mathbb{T}^{2*}M} \mathbb{T}^* \overset{2}{\mathbb{T}}^*M \otimes \mathbb{T}^* \overset{2}{\mathbb{T}}^*M,$$

$\overset{2}{\omega}_M$ can be viewed as a 'bi-form' or a pair of symplectic forms on the first and the second factor of \mathbb{T}^*M . This establishes a connection between the mapping β and the poli-symplectic formalism of Günther ([12]).

6 Example

Since our model is a very simple and designed to study geometrical structures related to the Classical Field Theory rather than to describe real physical systems, it is not easy to find physically important examples. In the literature, one can find the so called *bosonic string theory*. There are two approaches to the subject. In one of them, due to Polyakov [23], configurations are mappings from a two-dimensional manifold X of the string into the product of Minkowski space and the space of symmetric tensors on X . It means that not only the space-time position of the string is subject to variations, but also the metric on the string itself. In the simpler approach by Nambu [11, 21], the metric on the string is fixed to be the pull-back of the Minkowski metric by the string space-time configuration. In the Nambu approach we deal therefore with mappings from the two-dimensional manifold to the Minkowski space. In our example we will use the Nambu version with another simplification by taking $X = \mathbb{R}^2$.

The Minkowski space (M, V, η) is a four-dimensional affine space with the model vector space V equipped with a bilinear symmetric form η of signature $(+ - - -)$. We will denote by $\tilde{\eta}$ the associated self-adjoint map from V to V^* . Using the affine structure, we can identify the tangent bundle $\tau_M : TM \rightarrow M$ with the trivial bundle $M \times V \rightarrow M$, and the cotangent bundle $\pi_M : T^*M \rightarrow M$ with the trivial bundle $M \times V^* \rightarrow M$. The spaces that appear in the Lagrangian picture are therefore

$$\begin{aligned} \overset{2}{T}M &= M \times V \times V, \\ \overset{2}{T}^*M &= M \times V^* \times V^*, \\ T^* \overset{2}{T}M &= (M \times V \times V) \times (V^* \times V^* \times V^*), \\ \overset{2}{T} \overset{2}{T}^*M &= (M \times V^* \times V^*) \times (V \times V^* \times V^*) \times (V \times V^* \times V^*). \end{aligned}$$

The first jet of a mapping $u : \mathbb{R}^2 \rightarrow M$ at the point (x^1, x^2) is identified with a triple (q, v_1, v_2) , where $q = u(x^1, x^2)$, v_1 is a vector tangent to the curve $t \mapsto u(x^1 + t, x^2)$ at $t = 0$, and v_2 is a vector tangent to the curve $t \mapsto u(x^1, x^2 + t)$ at $t = 0$. The Lagrangian at the point $j^1 u$ is the scalar density associated to $u^* \eta$ which (after identification with the function on $M \times V \times V$) gives

$$(6.1) \quad L(q, v_1, v_2) = \sqrt{-\det g},$$

where

$$g = \begin{bmatrix} \eta(v_1, v_1) & \eta(v_1, v_2) \\ \eta(v_1, v_2) & \eta(v_2, v_2) \end{bmatrix}.$$

The Lagrangian is defined on the open set of $M \times V \times V$, where the determinant of the matrix g is negative. Denoting with (q, p^1, p^2) an element of $\overset{2}{T}^*M = M \times V^* \times V^*$, and with $(q, p^1, p^2, v_1, p^1_1, p^2_1, v_2, p^1_2, p^2_2)$ an element of $\overset{2}{T} \overset{2}{T}^*M$, we obtain the phase equations

$$(6.2) \quad v_1 = \frac{dq}{dx_1}, \quad v_2 = \frac{dq}{dx_2},$$

$$(6.3) \quad p^1 = \frac{1}{\sqrt{-\det g}} [\eta(v_1, v_2) \tilde{\eta}(v_2) - \eta(v_2, v_2) \tilde{\eta}(v_1)],$$

$$(6.4) \quad p^2 = \frac{1}{\sqrt{-\det g}} [\eta(v_1, v_2) \tilde{\eta}(v_1) - \eta(v_1, v_1) \tilde{\eta}(v_2)],$$

$$(6.5) \quad p^1_1 + p^2_2 = 0.$$

The Legendre map is in our example reversible, therefore we can express infinitesimal configurations in terms of momenta:

$$(6.6) \quad v_1 = \frac{-1}{\sqrt{-\det g}} [\eta(p^1, p^2) \tilde{\eta}^{-1}(p^2) - \eta(p^2, p^2) \tilde{\eta}^{-1}(p^1)],$$

$$(6.7) \quad v_2 = \frac{-1}{\sqrt{-\det g}} [\eta(p^1, p^2) \tilde{\eta}^{-1}(p^1) - \eta(p^1, p^1) \tilde{\eta}^{-1}(p^2)].$$

In the above formulae we used the same letter η for the bilinear form associated to η on the dual side. The matrix g , in terms of momenta, takes the form

$$g = \begin{bmatrix} -\eta(p^2, p^2) & \eta(p^1, p^2) \\ \eta(p^1, p^2) & -\eta(p^1, p^1) \end{bmatrix}.$$

Starting from the Hamiltonian

$$H(q, p^1, p^2) = -\sqrt{-\det g},$$

we obtain the phase equations of the form

$$(6.8) \quad \frac{dq}{dx^1} = \frac{1}{\sqrt{-\det g}} [\eta(p^1, p^2) \tilde{\eta}^{-1}(p^2) - \eta(p^2, p^2) \tilde{\eta}^{-1}(p^1)],$$

$$(6.9) \quad \frac{dq}{dx^2} = \frac{1}{\sqrt{-\det g}} [\eta(p^1, p^2) \tilde{\eta}^{-1}(p^1) - \eta(p^1, p^1) \tilde{\eta}^{-1}(p^2)],$$

$$(6.10) \quad \frac{dp^1}{dx^1} + \frac{dp^2}{dx^2} = 0,$$

which are of course the same as the phase equations generated by the Lagrangian description of the system. Let us finish this example with writing down the fundamental maps α and β : On the Lagrangian side we have

$$\mathsf{T}^* \overset{2}{\mathsf{T}} M = (M \times V \times V) \times (V^* \times V^* \times V^*),$$

so that

$$\alpha(q, p^1, p^2, v_1, p^1_1, p^2_1, v_2, p^1_2, p^2_2) = (q, v_1, v_2, p^1_1 + p^2_2, p^1, p^2).$$

On the Hamiltonian side, if we identify $\mathsf{T}^* \overset{2}{\mathsf{T}} M$ with $(M \times V^* \times V^*) \times (V^* \times V \times V)$, we obtain

$$\beta(q, p^1, p^2, v_1, p^1_1, p^2_1, v_2, p^1_2, p^2_2) = (q, p^1, p^2, -p^1_1 - p^2_2, v_1, v_2).$$

7 Conclusions

We have presented a toy model of a Classical Field Theory to introduce main concepts of a new approach to Lagrange and Hamilton formalisms. The starting point was the Tulczyjew triple in the Classical Mechanics, generalized now to the case of fields. In this approach all main ingredients are present: starting with a Lagrangian, not only the Euler-Lagrange field equation has been derived, but also the phase space and phase dynamics have been recognized, together with the Legendre map and the Hamiltonian picture. The latter suggests that momenta are dual rather to infinitesimal variations (displacements) than to infinitesimal configurations ('velocities'). The main difference with respect to the classical situation is that, to construct the phase dynamics, relations are used instead of mappings. This approach, presented here for maps from the disc into a manifold, can be naturally generalized to sections of a fibration and to an 'algebroid' setting as well. We postpone these studies to a separate paper.

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